Stability analysis of gravity-driven infiltrating flow

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[1] Stability analysis of gravity-driven unsaturated flow is examined for the general case of Darcian flow with a generalized nonequilibrium capillary pressure-saturation relation. With this nonequilibrium relation the governing equation is referred to as the nonequilibrium Richards equation (NERE). For the special case where the nonequilibrium vanishes, the NERE reduces to the Richards equation (RE), the conventional governing equation for describing unsaturated flow. A generalized linear stability analysis of the RE shows that this equation is unconditionally stable and therefore not able to produce gravity-driven unstable flows for infinitesimal perturbations to the flow field. A much stronger result of unconditional stability for the RE is derived using a nonlinear stability analysis applicable to the general case of heterogeneous porous media. For the general case of the NERE model, results of a linear stability analysis show that the NERE model is conditionally stable, with lower-frequency perturbations being unstable. A result of this analysis is that the nonmonotonicity of the pressure and saturation profile is a requisite condition for flow instability. *INDEX TERMS:* 1875 Hydrology: Unsaturated zone; 1866 Hydrology: Soil moisture; 3210 Mathematical Geophysics: Modeling; *KEYWORDS:* Richards' equation, gravity-driven flow, dynamic capillary pressure, traveling wave solution, stability analysis

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1. Introduction

[2] Laboratory and field observations [*Lissey*, 1971; *Glass et al.*, 1989; *Kung*, 1990; *Isensee et al.*, 1990; *Prazak et al.*, 1992; *Ritsema et al.*, 1993; *Shalit et al.*, 1995; *Chakka and Munster*, 1997; *Elliott et al.*, 1998] and conceptual models [*Roth*, 1995; *Nieber*, 1996; *Ju and Kung*, 1997; *Jarvis*, 1998] of vadose zone processes have pointed to the significance of preferential flow phenomena on the rapid flow of water and transport of contaminants through the vadose zone. *Nieber* [2000] discusses several types of preferential flow processes, with gravity-driven unstable flow being one of them. The experimental and mathematical study of gravity-driven unstable flow over the past three decades has led to the conclusion that unstable flows can occur over the wide range of conditions that are found under field conditions.

[3] The practical importance of the fingering phenomenon has motivated much effort over the last three decades [*Raats*, 1973; *Philip*, 1975; *Diment et al.*, 1982; *Hillel and Baker*, 1988; *Glass et al.*, 1989; *Baker and Hillel*, 1990; *Nieber*, 1996; *Ursino*, 2000; *Du et al.*, 2001; *Eliassi and Glass*, 2001; *Dautov et al.*, 2002; *Egorov et al.*, 2002; *Nieber et al.*, 2003] to better understand this phenomenon from physical and mathematical standpoints. While the studies that have been performed have each provided an understanding of some specific aspects of the fingering phenomenon, we conclude from our findings that an overall physically-based description remains to be discovered. On the basis of previous work and our results to date we conclude that to describe fingering, a mathematical model must bear at least two principal features: (1) the model must be able to generate initial unstable growth of small perturbations of the wetting front, and (2) it must be able to promote persistence of the initially growing perturbations by limiting lateral spreading behind the unstable front. This manuscript addresses the first of these two features. The second feature will be considered in a subsequent manuscript on two-dimensional flow.

[4] As mentioned above, the conditions that initiate unstable flows are not well understood. With respect to the Richards equation (RE), it being the conventional equation to model unsaturated flow in porous media, we can declare based on the nonlinear stability analysis of *Otto* [1997] that the RE model cannot produce instability of a wetting front in homogeneous porous media and therefore cannot be used as a model for finger flow in homogeneous porous media. This conclusion is derived from the stability estimates of standard boundary value problems for the Richards equation by *Alt et al.* [1984] and *Otto* [1997], and is valid for any applied finite perturbation. This result is

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mathematically accurate and stronger than the one derived from linear stability analysis which deals with infinitesimal perturbations. One limitation of the analysis by *Otto* [1997] is that it considered the porous medium to be homogeneous. In section 3 of this manuscript, Otto's conclusion about the unconditional stability of the RE are extended to the case of heterogeneous porous media.

[5] It is astonishing that despite the long history of investigating stability of the RE model with linear stability analysis, mathematically accurate results have been obtained only for the limiting case of the Green and Ampt model by *Raats* [1973] and *Philip* [1975]. The likely reason is that the spectral problem for the resulting non-self-adjoint second order differential equation in infinite domains is very complicated. Therefore, in all of the theoretical studies known to us, authors have made some simplifying assumptions, which have led the authors to draw conclusions about the stability of the RE which are either not generally applicable [*Diment and Watson*, 1983; *Ursino*, 2000] or are incorrect [*Kapoor*, 1996; *Du et al.*, 2001].

[6] The most thorough analysis of the stability of Richards equation was performed by *Diment et al.* [1982] and *Diment and Watson* [1983]. *Diment et al.* [1982] derived the differential equation that describes the perturbation problem and examined it numerically for a limited number of cases [*Diment and Watson*, 1983]. They found that the traveling wave solution, being a basic solution for the Richards equation as derived by *Philip* [1957], is stable. However, numerical solutions, even performed very accurately, cannot provide the stability criteria, while a general analytical study will do so. Therefore we were motivated to provide an analytical linear stability analysis of the RE in spite of the general results of *Otto* [1997].

[7] In section 4 we provide this linear stability analysis of the Philip's traveling wave solution of the RE [*Philip*, 1957] for the process of gravity-driven infiltration. We note that in the analysis we do not make any simplifying assumptions. By use of the two main ideas of (1) symmetrization of the perturbed equation and (2) the invariant properties of the basic solution, the analysis may be done analytically.

[8] A second motivation for presenting the general linear stability analysis of the RE is to apply the analysis method to modifications of the RE to identify the types of modifications needed to admit flow instabilities. Therefore techniques developed in section 4 are then used in sections 5 and 6 for stability analysis of the traveling wave solution for two other models of flow in unsaturated porous media. The first model, called here the sharp front Richards equation (SFRE), based on the assumption of air entry pressure [*Hillel and Baker*, 1988] was introduced by *Selker et al.* [1992] to model the moisture profile within a finger. The SFRE model dictates that there is a sharp wetting front where water saturation jumps from an initial value beyond the front to the value related to the air entry pressure, and the Richards equation holds behind the front. In section 5 we conclude that the traveling wave solution for the SFRE model is unstable, and the higher the frequency of the lateral perturbation, the faster the growth of the corresponding perturbation with time.

[9] The second model, introduced in section 6, is the nonequilibrium Richards equation (NERE) accounting for a general nonequilibrium capillary pressure-saturation relationship. To obtain the NERE we hold the standard water mass balance equation, and replace the equilibrium pressure-saturation relation with a general nonequilibrium relation. Introducing several assumptions to the class of such modifications, we study the low-frequency behavior of the perturbed flow equation and derive the stability criterion. This criterion relates to the nonmonotonicity of the pressure profile for the basic traveling wave solution. As a conclusion, we treat the nonmonotonicity of the traveling wave solution as a necessary condition to the hypothetical model for unstable flow in unsaturated porous media.

[10] The results on stability for all models are discussed in section 7. These results are also compared to results from previous studies.

[11] Within the presentation on flow stability the following terminology adopted from the theory of stability [*Joseph*, 1976] will be used. Asymptotic stability means that perturbations will decay with time. Neutral stability means that perturbations will neither decay, or grow with time. Instability means that perturbations will grow without limit with time. Throughout most of the manuscript when we mean asymptotic stability we will use the term stability or stable.

2. Richards Equation

[12] The conventional Richards equation (RE) for the flow of water in unsaturated porous media may be written in dimensionless form as

$$\frac{\partial s}{\partial t} - \nabla \cdot K(s)\nabla p + \frac{\partial}{\partial z}K(s) = 0, \tag{1}$$

$$p = \mathcal{P}(s), \tag{2}$$

where s is the effective saturation ($0 \le s \le 1$), p is the water pressure, K is the relative hydraulic conductivity, and z is the vertical coordinate taken positive downward. Function $\mathcal{P}$ is the equilibrium pressure being a function of s (solid line in Figure 1). This function monotonically increases with s from $-\infty$ at $s = 0$ to a some limiting value taken zero at $s = 1$. To have the RE work in the saturated region the function $\mathcal{P}(s)$ prolongs by a vertical line $0 \le p < \infty$ at $s = 1$ and relation (2) may be treated as an inclusion meaning that p belongs to a graph of the corresponding multivalued function.

[13] With the approach frequently used for saturated-unsaturated porous media, we introduce the single-valued function $\mathcal{S}$, the inverse to $\mathcal{P}$, by rewriting equation (2) as

$$s = \mathcal{S}(p), \tag{3}$$

and solve the system equations (1) and (3) for pressure as a primary variable. Equation (3) will be applied to the nonlinear stability analysis of the RE presented in section 3.

[14] Limiting the analysis to the unsaturated regions the system equations (1) and (2) may be reduced to one equation with s being a primary variable by introducing the diffusivity function $D(s) = K(s)\mathcal{P}'(s)$ (prime indicates the first derivative with respect to saturation s):

$$\frac{\partial s}{\partial t} - \nabla \cdot D(s)\nabla s + \frac{\partial}{\partial z}K(s) = 0. \tag{4}$$

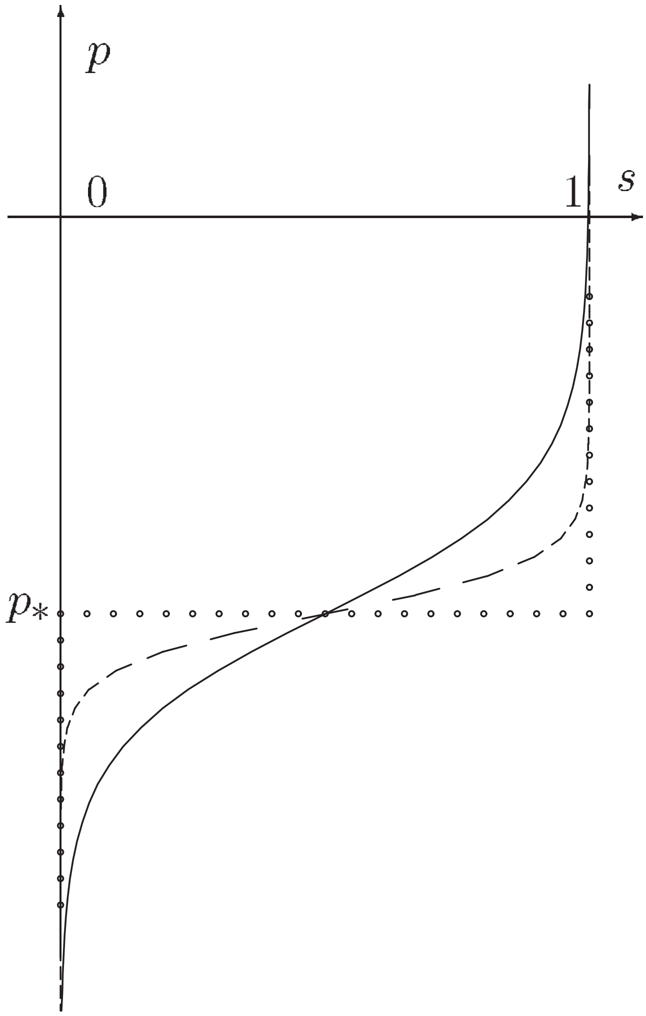

Figure 1. Capillary pressure-saturation function for Richards equation (solid line), Green-Ampt model (dotted line), and intermediate curve $p = p_* + \chi \cdot (\mathcal{P}(s) - p_*)$ (dashed line).

In this paper we require that functions $K(s)$, $D(s)$ and $\mathcal{P}(s)$ bear the following standard properties within the interval (0,1): they are monotonically increasing and continuously differentiable functions, and $K(s)$ is strictly convex, such as $K(1) = 1$, $K(s) \propto s^{\alpha}$, $D(s) \propto s^{\beta}$ as $s \to 0$, $\alpha > 1$, $\beta > 0$. Equation (4) will be applied to the linear stability analysis given in sections 4.1 and 4.2.

3. Stability Analysis of the RE Model: General Results

[15] The RE model expressed in the form of equations (1) and (3) belongs to a wide class of nonlinear elliptic partial differential equations analyzed by *Alt et al.* [1984] and *Otto* [1997]. Using a Kirchhoff transformation

$$u = \int_0^p K(\mathcal{S}(p))dp$$

reduces the system equations (1) and (3) to

$$\frac{\partial s}{\partial t} - \nabla \cdot \nabla u + \frac{\partial}{\partial z} K(s) = 0, \quad s = b(u), \quad b(u) = \mathcal{S}(p(u)).$$

For a bounded piecewise smooth region $\Omega = (x,y,z)$ and standard boundary conditions, *Alt et al.* [1984, theorem 1.7] show the existence of the weak solution of "finite energy" for this problem. Guided by their result, *Otto* [1996, 1997] proved uniqueness of the solution and the stability estimate of the form

$$\int_{\Omega} |s_1(t) - s_2(t)| d\Omega \leq \int_{\Omega} |s_1^0 - s_2^0| d\Omega \tag{5}$$

for any two given solutions s_1 and s_2 corresponding to initial conditions with $s_1{}^0$ and $s_2{}^0$ respectively. Assumptions made regarding the behavior of the functions b: $\mathbb{R} \to [0, 1]$ and K: $[0, 1] \to [0, 1]$, are relatively unrestrictive and are satisfied for functions $K(s)$, $D(s)$, and $\mathcal{P}(s)$ that are restricted by the assumptions given above. Inequality equation (5) dictates that if the saturation field $s_1{}^0$ at the initial time is arbitrarily perturbed by $\delta^0 = s_2{}^0 - s_1{}^0$, then the perturbation $\delta = s_2 - s_1$ thereafter will not grow and always be less than the initial one ($\|\delta\| \leq \|\delta^0\|$).

[16] The analysis provided by *Otto* [1997] is complicated. The reason for this is because Otto studied a general case allowing dry zones with $s = 0$ to exist in the solution. If there are no dry zones assumed at the initial time than it is obvious from a physical standpoint that they will not appear within a finite time. If this assumption is valid a priori, then the proof of equation (5) can be achieved more easily than in the work by *Otto* [1997] by using the standard technique given by *Alt et al.* [1984]. This technique does not utilize the Kirchhoff transformation, which is limited to homogeneous porous media, and therefore may be applied to the general case of heterogeneous porous media. The presentation of this technique is given in Appendix A in which it is shown that the RE is stable even for the case of heterogeneous porous media.

[17] Referring to the terminology of stable/unstable solutions given in the Introduction, we can state that for asymptotically stable and for neutrally stable solutions $\|\delta\|$ is limited regardless of the form of the initial perturbation δ^0, and $\|\delta\| \to 0$ as $t \to \infty$ for asymptotically stable solutions. While for unstable solutions $\|\delta\| \to \infty$ as $t \to \infty$ for some initial perturbation δ^0. Holding to this terminology we declare that inequality equation (5) points to the conditions of either asymptotic, or neutral stability for any solution of the RE. For the general situation considered by *Otto* [1997] it is impossible to specify the type of stability, either asymptotic or neutral, but the type will depend on the form of both the basic solution and boundary conditions. For instance, for the case $s_1{}^0 \equiv 0$ and a no flow boundary condition we obtain from the mass balance equation for any initial perturbation δ^0 that $\|\delta\| = \|\delta^0\|$, and hence the trivial zero solution is neutrally stable. On the other hand, if the boundary conditions are taken such as they lead to a steady state solution then the solution for any initial conditions tends to this steady state at $t \to \infty$, and hence any solution for these boundary conditions is asymptotically stable.

[18] The conclusion of the discussion above is that the RE model is stable for any perturbation for either homogeneous or heterogeneous porous media, and therefore it makes the

RE model inappropriate to model fingering in the unsaturated porous media.

[19] In the literature on soil hydrology and soil physics [*Diment et al.*, 1982; *Kapoor*, 1996; *Ursino*, 2000; *Du et al.*, 2001] stability of the RE has been studied by the linear stability approach applying infinitesimal initial perturbation δ^0 to the basic solution. Unlike the general results on stability discussed in this section, the principle of the linear stability guarantees stability of the basic solution only for infinitesimal perturbations. This principle is much weaker than principle (5) of global stability provided by nonlinear theory, since global stability includes linear stability but not vice versa. The real value of linear stability analysis is found in its ability to manifest conditions under which the basic solution might possibly be unstable.

[20] In the remainder of this paper linear stability analysis of various flow equations will be analyzed. This analysis will present analytical results that have not been presented previously for the RE and will reveal the modifications to the RE that are necessary to allow instability of flow.

4. Stability Analysis of the RE Model: Linear Analysis

[21] Following the conventional approach of linear stability analysis, the process of wetting front instability will be examined by superposition of a perturbed flow regime onto a basic flow regime. The basis flow is derived through the traveling wave solution of the one-dimensional Richards equation as described by *Philip* [1957] and *Parlange* [1971]. The perturbed flow is derived through a local linearization of the Richards equation in three dimensions. Mathematical analysis (analytical or numerical) for this linear equation must be done to predict whether a perturbation will grow or decay. If the perturbation decreases with time, the wetting front is stable. If the perturbation increases with time, the wetting front is unstable. Such procedure is used in the stability analysis presented below.

4.1. Basic Solution

[22] We consider a typical example of fingering when water is applied to an upper surface of the initially "dry" unsaturated porous medium $z \geq 0$. The water flux q is less than the saturated hydraulic conductivity of the porous medium. For this case, one-dimensional solution of the Richards equation (1) and (2) represents a gravity-driven wetting wave and the solution approaches the self-similar traveling wave-type regime with time. This traveling wave solution is used as a basic solution to provide a stability analysis of the gravity-driven infiltrating flow.

[23] The traveling wave solution for equations (1) and (2) with the traveling wave coordinate ξ

$$s = s_0(\xi), \quad \xi = z - vt \tag{6}$$

was developed by *Philip* [1957]. The boundary conditions

$$s_0(-\infty) = s_-, \quad s_0(+\infty) = s_+, \quad 0 < s_+ < s_- < 1 \tag{7}$$

specify values of the saturation ahead (s_+) of the wetting front (in the "dry" region) and behind (s_-) the wetting front (in the "wet" region), while the velocity of the wetting front v is defined as

$$v = \frac{K(s_+) - K(s_-)}{s_+ - s_-}. \tag{8}$$

The value of s_- is specified by the flux Q, and $Q = K(s_-)$.

[24] Substituting equation (6) into equation (4) and integrating the resulting ordinary differential equation with equation (7) results in

$$-D(s)\frac{ds_0}{d\xi} + K(s_0) - vs_0 = K(s_+) - vs_+. \tag{9}$$

Integrating both sides of equation (9) we obtain

$$\xi - \xi_* = \int_{s_0}^{s_*} \frac{D(s)ds}{v(s - s_+) - K(s) + K(s_+)}, \tag{10}$$

where s is the dummy variable, and s_* ($s_+ < s_* < s_-$) is the saturation at the arbitrary point ξ_*. Arbitrariness of ξ_* indicates that the Philip's solution (10) is valid with any spatial shift. We also emphasize that the denominator in equation (10) is positive for $s \in (s_+, s_-)$ and equal to zero at both ends $s = s_+$ and $s = s_-$, because $K(s)$ is considered to be a convex function. As a result, $s_0(\xi)$ monotonically decreases from s_- at $\xi = -\infty$ to s_+ at $\xi = +\infty$ (solid line in Figure 2), and has exponential asymptotic behavior at infinity:

$$\xi \to \pm\infty : \quad (s_0 - s_\pm) \propto \exp(\alpha_\pm \xi), \qquad \alpha_\pm = (K'(s_\pm) - v)/D(s_\pm). \tag{11}$$

4.2. Perturbed Flow Equation

[25] An essential requirement for stability analysis is a knowledge of the time response of the system to small disturbances. This is examined mathematically by assuming that the physical variables experience some small disturbance:

$$s = s_0(\xi) + \varepsilon s_1(x, y, \xi, t) + O(\varepsilon^2), \tag{12}$$

where $s(x, y, \xi, t)$ is the perturbed saturation field, s_0 is the basic solution (10), s_1 is the bounded perturbation, and ε is a small constant representing the size of the disturbance. For convenience, in this analysis we use saturation as a primary function and moving coordinate $\xi = z - vt$ as a primary variable instead of pressure and z respectively as used by *Diment et al.* [1982], *Kapoor* [1996], and *Ursino* [2000].

[26] The Richards equation is assumed to hold not only for the basic solution but also for the perturbed solution. Therefore we (1) rewrite the Richards equation in moving coordinates (x, y, ξ, t) as

$$\frac{\partial s}{\partial t} + \frac{\partial}{\partial \xi}(K(s) - vs) - \nabla \cdot (D(s)\nabla s) = 0, \tag{13}$$

where $\nabla = (\partial_x, \partial_y, \partial_\xi)$, (2) substitute the form of the solution given by equation (12) into governing equation (13), and (3) linearize the resulting expression ignoring quadratic and

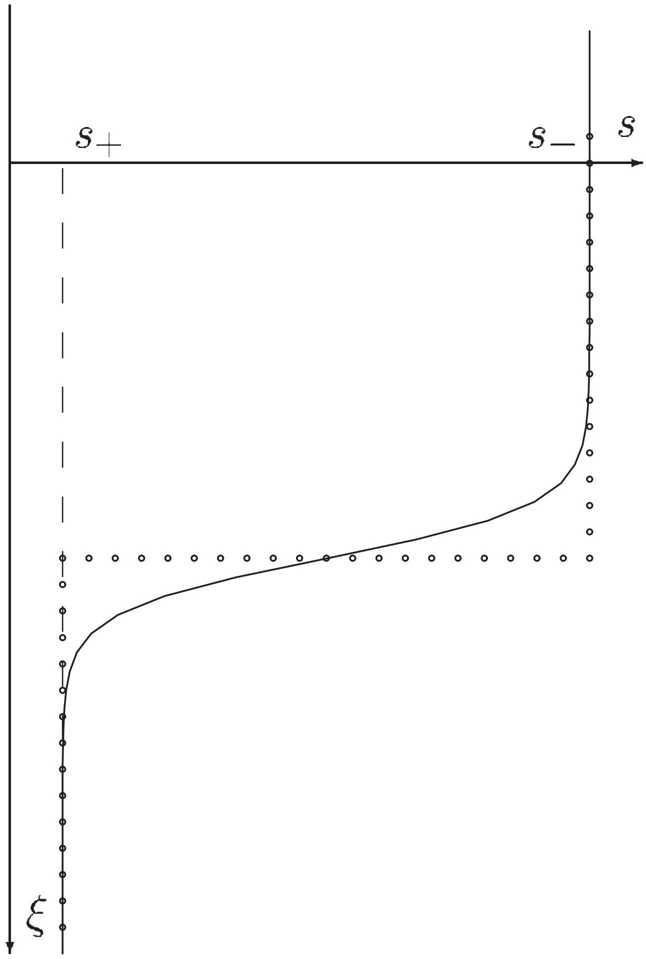

Figure 2. Saturation profile of traveling wave solution of the Richards equation (solid line) and Green-Ampt model (dotted line).

higher-order terms. Combining terms with zero and first power of ε and equating them to zero leads to two equations. The first equation has terms with ε^0, and solution to this equation known as a basic solution is presented by equation (10). Terms with ε^1 occur in the second equation. This equation describes the behavior of the imposed disturbance and it referred to as the first-order linear perturbation equation, and is given by

$$\frac{\partial s_1}{\partial t} + \frac{\partial}{\partial \xi}\left[\left(K'(s_0) - v - D'(s_0)\frac{ds_0}{d\xi}\right)s_1\right] - \nabla \cdot (D(s_0)\nabla s_1) = 0, \quad (14)$$

where $D'(s)$ and $K'(s)$ are the first derivatives of $D(s)$ and $K(s)$ with respect to saturation s. The initial condition describes an initial form of the perturbation:

$$t = 0: \quad s_1 = s_{\text{init}}(x, y, \xi). \quad (15)$$

Solution of the formulated Cauchy problem (14) with initial condition (15) describes the evolution of an arbitrary perturbation with compact support, $\varepsilon s_{\text{init}}$, imposed into the basic solution at the initial time.

[27] In the following the general disturbance of the wetting front is given by the sum of a number of basic modes through a Fourier series representation, and the instability of the general disturbance is then determined by stability analysis of the basic modes. This spectral analysis is conducted by transforming equation (14) to the frequency domain via a Fourier transformation. With this transformed equation the stability of individual Fourier modes (disturbance wavelengths) can be assessed. The Fourier transform of equation (14) taken with respect to x and y leads to

$$\frac{\partial \bar{s}_1}{\partial t} - \frac{\partial^2}{\partial \xi^2}(D(s_0)\bar{s}_1) + \frac{\partial}{\partial \xi}((K'(s_0) - v)\bar{s}_1) + \omega^2 D(s_0)\bar{s}_1 = 0, \quad (16)$$

where $\omega(\omega^2 = \omega_x{}^2 + \omega_y^2)$ is the wave number, ω_x and ω_y are the angular frequencies in the Fourier transform of s_1 with respect to x and y respectively, and $\bar{s}_1$ is the Fourier transform of s_1. Since the superposition principle is valid for the linear problem given by equations (14) and (15), the condition (15) can be equally imposed on equation (16). It is necessary to only replace in equation (15) the functions s_1 and s_{init} by Fourier transforms $\bar{s}_1$ and $\bar{s}_{\text{init}}$ respectively.

[28] The form of the perturbation equation (16) is complicated and seems to be inappropriate for analytical study, and therefore it needs to be modified to be more tractable. For this modification we symmetrize this equation by replacing $\bar{s}_1$ with a new function θ, and ξ with a new coordinate ζ ($-\infty < \zeta < +\infty$):

$$\zeta = \int \frac{d\xi}{\sqrt{D(s_0)}}, \quad \theta = \frac{D^{1/4}(s_0)}{\sqrt{-s_0'}}\bar{s}_1. \quad (17)$$

Identical transformations provided in Appendix B reduce the problem (16) to the Cauchy problem (18) for $\theta(\zeta, t)$:

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial \zeta^2} - \left(\omega^2 D + F\right)\theta, \quad \theta(\zeta, 0) = \theta_{\text{init}}(\zeta), \quad (18)$$

$$F = \frac{1}{B}\frac{d^2 B}{d\zeta^2}, \quad B = D^{1/4}\left(-s_0'\right)^{1/2}, \quad (19)$$

where function θ_{init} has a compact support, function $D(\zeta)$ monotonically decreases with ζ from $D_- = D(s_-)$ at $\zeta = -\infty$ to $D_+ = D(s_+)$ at $\zeta = +\infty$, function $B(\zeta)$ exponentially approaches zero at infinity, and $F(\zeta)$ has upper and lower limits as illustrated in Figure 3 and exponentially approaches finite positive values as $\zeta \to \pm\infty$:

$$F(\pm\infty) = F_\pm, \quad F_\pm = \frac{\left(K'_\pm - v\right)^2}{4D_\pm}, \quad K'_\pm = K'(s_\pm). \quad (20)$$

We notice that the formulae in equation (20) are derived from equation (19) with known behavior of s_0 at infinity defined in equation (11).

4.3. Stability Analysis

[29] Coefficients D and F are independent of time t, and separating variables we attempt to solve the Cauchy prob-

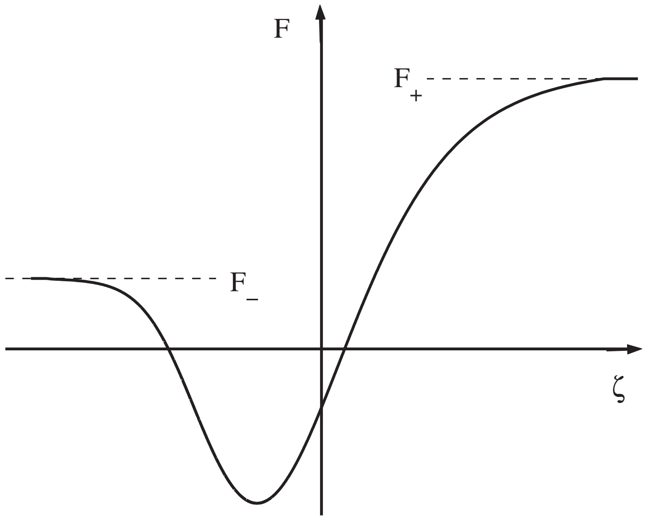

Figure 3. Typical profile of $F(\zeta)$.

lem (18) seeking a spectrum of the operator appearing in the right-hand side of equation (18). In other words, we seek all values of k such that the equation

$$\frac{d^2\Psi}{d\zeta^2} - \left(\omega^2 D + F\right)\Psi = -k\Psi, \quad -\infty < \zeta < \infty \qquad (21)$$

has bounded and nonzero solutions $\Psi(\zeta;\, k)$. In this case, the solution of the Cauchy problem (18) represents a linear superposition of terms $\Psi(\zeta;\, k)\ \exp\ (-kt)$ (in general a Stieltjes' integral). It is apparent that the solution damps out with time if the spectrum lies in the semi-plane $\mathrm{Re}(k) > 0$ of the complex plane k. Therefore stability of the traveling wave solution relates to positivity of the real part of the spectrum of the problem (21).

[30] Thus we have arrived at the classical mathematical problem (21) known as Sturm-Liouville problem. This problem is singular because ζ is defined within an infinite domain. In quantum mechanics equation (21) is called the Schrödinger equation. Having similarity with the Schrödinger equation we may say that it determines a stationary state of particles with energy k in external potential field $U = \omega^2 D + F$. This problem being very important in practice has been thoroughly studied and properties of the solution are established for various forms of the potential U [*Carmona and Lacroix*, 1999].

[31] For the problem of interest to us, the potential U is uniformly bounded from below for any ζ, and hence, the operator of the problem is self-adjoint and semi-bounded from below in the space $L_2(-\infty,\, \infty)$ of functions such as their quadratics are integrated over the entire axis. Therefore the spectrum of the operator is real and two parts of the spectrum may be distinguished: the continuous part for $k \in [k_{\mathrm{ess}}(\omega),\, \infty)$, and the discrete part $k_0(\omega) < k_1(\omega) < \ldots < k_m(\omega) < k_{\mathrm{ess}}(\omega)$ (perhaps empty) between a minimum value of U and the lowest point $k_{\mathrm{ess}}(\omega)$ of the continuous spectrum ($m < \infty$). Points k of the continuous spectrum correspond to bounded solutions $\Psi(\zeta;\, k)$ of the problem (21) (generalized eigenfunctions), while eigenvalues $k_i(\omega)$, $i = 0, \ldots, m$ correspond to the functions $\Psi(\zeta;\, k_i(\omega)) \in L_2(-\infty,\, \infty)$ decreasing as ζ tends to infinity. The value of $k_{\mathrm{ess}}(\omega)$ is determined only by asymptotic behavior of the potential U at infinity

$$k_{\mathrm{ess}}(\omega) = \min(U(-\infty), U(+\infty)) = \\ = \min\left(\omega^2 D_- + F_-, \omega^2 D_+ + F_+\right). \qquad (22)$$

$k_{\mathrm{ess}}(\omega)$ is positive due to positivity of both $D_\pm$ and $F_\pm$. At last, to establish that the traveling wave solution for the Richards equation is stable we need to prove that if the smallest eigenvalue k_0 exists, that it is positive.

[32] We start the analysis with the limiting case $\omega = 0$ and utilize the approach of *Barenblatt* [1996, p. 212]. By trial-and-error setting the eigenvalue $k = 0$ we found that B satisfies equation (21), and therefore $k = 0$ and function B are eigenvalue and eigenfunction respectively for this case. Function B given by the definition (19) has no zeroes for finite ζ. However, the eigenfunction should have as many zeros as its ordinal number that forces B to correspond to the smallest eigenvalue. Since $k = 0$ there are no negative eigenvalues in the problem, and $k_0 = 0$ for $\omega = 0$.

[33] For a nonzero wave number $\omega > 0$ the smallest eigenvalue $k_0(\omega)$ obeys the inequality

$$D_+\omega^2 \le k_0(\omega) \le D_-\omega^2. \qquad (23)$$

To prove (23) we use the variational principle from [chap. 13.1 *Reed and Simon*, 1978] adapted to our problem:

$$k_0(\omega) = \min_{\psi}\left(\int_{-\infty}^{\infty}\left(\psi'^2 + \left(\omega^2 D + F\right)\psi^2\right)d\zeta \Big/ \int_{-\infty}^{\infty}\psi^2 d\zeta\right), \quad (24)$$

where the minimum is taken over such functions that squares of both functions and their first derivatives are integrable. Using the usual technique [*Reed and Simon*, 1978] and noting that the minimum of the sum of functions is not less than the sum of the minimums of functions yields from (24)

$$k_0(\omega) \ge k_0(0) + \omega^2 \min_{\psi}\left(\int_{-\infty}^{\infty} D\psi^2 d\zeta \Big/ \int_{-\infty}^{\infty}\psi^2 d\zeta\right) = D_+\omega^2.$$

This proves the left inequality in (23). The right inequality in (23) may be proved by substituting B as a test function ψ in equation (24). With this proof and knowing that D_+ is positive we establish that for any $\omega > 0$ the spectrum of the problem (21) is positive. This immediately proves that the traveling wave solution of the Richards equation is stable.

[34] It is interesting to investigate the behavior of the smallest eigenvalue k_0 for the problem (21) as a function of ω. This function relates to how the potential $U(\zeta;\omega)$ behaves with a change of ω. For a small ω the profile of $U(\zeta;\omega)$ resembles the form of $F(\zeta)$ and has a typical region of depression as demonstrated in Figure 3. The preceding analysis showed that the minimum value in this region is sufficiently small so the eigenvalue k_0 exists for the problem (21). Increasing ω, the curve $U(\zeta)$ shifts upward at all points ζ since D is positive, and it causes a monotonic growth of $k_0(\omega)$ (see (24)). The depth of the potential depression region diminishes with increase of ω since D is monotonic and the depression region eventually disappears at some

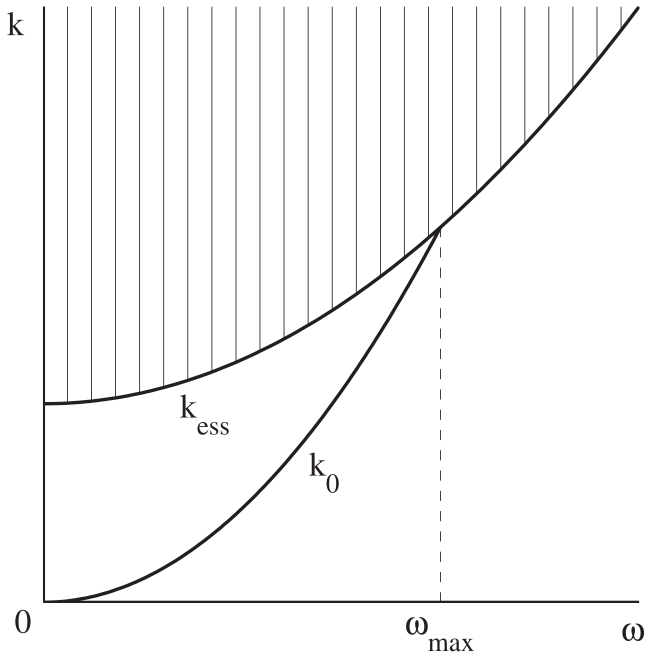

Figure 4. Spectrum of the problem (21). Continuous spectrum occupies the hatched domain.

value of ω making U a monotonic function of ζ. For this case the problem (21) cannot have eigenvalues and the whole spectrum is continuous. So there exists a maximum limiting value ω_{max} such that the smallest eigenvalue approaches the border of the continuous spectrum as shown in Figure 4.

[35] We should reiterate the fact that we implemented a standard hydrodynamic stability analysis approach to prove that the traveling wave solution of the Richards equation is stable. Such approach a priori postulates that the perturbed saturation field s is represented in the form $\left(\mathrm{i} = \sqrt{-1}\right)$

$$s = s_0(\xi) + \varepsilon S(\xi)\exp\left(\mathrm{i}\left(w_x x + w_y y\right) - kt\right) \qquad (25)$$

that leads to a spectral problem for eigenfunctions S and eigenvalues k [*Diment et al.*, 1982] similar to the problem (22) and (24). The approach used in this paper differs from the standard approach because we sought function $\Psi = SD^{1/4}(-s_0')^{-1/2}$ instead of S and use variable ζ instead of ξ. Such transformations make the foregoing analysis of the spectral problem transparent.

[36] The special case $\omega = 0$ for a spectrum with no lateral perturbations should be discussed in some detail before finishing this analysis of the RE. According to equation (23), the main eigenvalue k_0 is equal to zero and, as result, the corresponding perturbation holds and does not damp out with time as it would be for the case $\omega > 0$. This would indicate, prima facie, that for this problem the flow will be unstable when $\omega = 0$. However, this is not a correct assessment according to the following explanation. We mentioned previously that the basic solution (10) is valid for any translation in the parameter ξ_*. For the case with $\omega = 0$, the eigenfunction for the perturbed solution $\Psi = B$ (or $S = s_0'$ in the standard perturbation analysis approach (25)) corresponds to a slight spatial shift of the basic solution on ε. Figure 8.4 of *Barenblatt* [1996] shows just such a perturbed solution and the shifted unperturbed solution to which the perturbed solution tends. This shifted solution is just the basic solution translated by a constant, and as such this shifted solution should not be considered an instability. As stated by Barenblatt, "If ... the perturbed solution tends not to the original unperturbed solution as $t \to \infty$, but to a shifted one, then there is no reason to consider this transition as an instability."

4.4. Transition to the Green and Ampt Model

[37] Mathematically accurate results for stability of the traveling wave solution had been obtained to date only for the relatively simple Green and Ampt model of infiltration [*Green and Ampt*, 1911]. The stability criterion first postulated by *Raats* [1973] and later improved and derived by *Philip* [1975] leads to a primitive result for the problem considered in this paper. According to Philip's criterion [*Philip*, 1975, p. 1044], "the physical attribute of the system which is fundamental to the question of stability is the water pressure gradient behind the wetting front. When this gradient assists the flow, the flow is stable; when this gradient opposes it, the flow is unstable." However, for the Green and Ampt model the pressure gradient behind the wetting front for the traveling wave type of the solution is equal to zero! This means that the traveling wave solution of the Green and Ampt model is neutral with regards to any frequency of lateral perturbations: perturbations neither grow nor damp out with time, and they hold constant such as the minimum eigenvalue k_0 in equation (25) is identically equal to zero for all ω.

[38] The preceding stability analysis may be also applied to the Green and Ampt model if small modifications are made. We generalize Richards equation by introducing a family of equilibrium pressure functions $\mathcal{P}_\chi(s) = p_* + \chi(\mathcal{P} - p_*)$ illustrated in Figure 1. It is evident that $\mathcal{P}_\chi = \mathcal{P}$ for $\chi = 1$, while $\mathcal{P}_\chi$ tends to a step function representing the Green and Ampt model for $\chi = 0$. Consequently, a diffusive front of the traveling wave solution of the Richards equation approaches a sharp front solution by the Green and Ampt model with decreasing of χ as shown in Figure 2.

[39] Replacing function $\mathcal{P}$ in equation (2) by $\mathcal{P}_\chi$ conserves the preceding analysis and final results but leads to small changes of the form of both the basic solution and the perturbed flow equation. In fact, the diffusivity function $D(s) = K\mathcal{P}'$ transforms to $\chi D(s)$. In the eigenvalue problem (21) coefficients k and ω^2 are replaced by χk and $\chi^2\omega^2$ respectively. These replacements show that if $k_0(\omega)$ represents a smallest eigenvalue for the Richards equation with $\chi = 1$, then $k_\chi = \chi^{-1}k_0(\chi\omega)$ represents a smallest eigenvalue for the Richards type equation with any $\chi > 0$. Then, equation (23) dictates that for any $\chi > 0$ the smallest eigenvalue k_χ is positive and tends to zero only as $\chi \to 0$. This confirms results known to date on stability of the Green and Ampt model, and the neutral traveling wave solution for the Green and Ampt model becomes asymptotically stable if any Richards type regularization is applied.

5. Sharp Front Richards Equation

[40] In the preceding sections we demonstrated that the traveling wave solution of the RE is unconditionally stable

and, hence, finger generation and breakup of the uniform wetting flow may not be described by this model. In this section we present a modification of the RE model, the sharp front Richards equation (SFRE) that might possess the capacity to describe this phenomenon. The SFRE model, is based on the concept of air entry pressure [*Hillel and Baker*, 1988] and was introduced by *Selker et al.* [1992]. The SFRE model is based on the assumption that the porous medium is divided by the sharp wetting front $z = z_e(x, y, t)$ into two regions: the porous medium remains initially dry with $s = s_+$ for $z > z_e$, and the RE is valid for $z < z_e$. Saturation s_e at the wetting side of the front is given by

$$z = z_e - 0: \quad s = s_e \tag{26}$$

and the value of s_e is close to unity and relates to the value of the air entry pressure p_e: $p_e = \mathcal{P}(s_e)$. An additional condition on the advancing front is mass conservation. This condition gives a relationship between the advancing front velocity and the difference of normal components of the flux at both sides of the front:

$$(s_e - s_+)\frac{\partial z_e}{\partial t} = K(s_e) - K(s_+) - D(s)\left(\frac{\partial s}{\partial z} - \frac{\partial z_e}{\partial x}\frac{\partial s}{\partial x} - \frac{\partial z_e}{\partial y}\frac{\partial s}{\partial y}\right)\Bigg|_{z=z_e-0}. \tag{27}$$

The traveling wave solution for the SFRE model is derived by *Selker et al.* [1992]. For the wetted region the solution has the same form as the Philip's solution (8) and (10), but now the values $s*$ and s_0 in (10) fall within the interval $[s_-, s_e]$ instead of $[s_+, s_-]$. The denominator in equation (10) is negative within $[s_-, s_e]$ and equal to zero at $s = s_-$. This means that the function $s_0(\xi)$ monotonically increases from s_- at $\xi = -\infty$ to s_e at $\xi = \xi_e^0$. s_0 approaches s_- as $\xi \to -\infty$ exponentially as in equation (11), and the advancing front coordinate ξ_e^0 is determined by equation (10) with $s_0 = s_e$. This basic solution is also invariant over the spatial shift along ξ and therefore we may set $\xi_e^0 = 0$.

5.1. Perturbed Flow Equation

[41] The preceding linear perturbation technique is applied. We derive the perturbation equation first and, then, formulate the problem on stability of the traveling wave solution for the SFRE model as a problem of seeking the spectrum of the perturbed equation. The perturbed saturation field and perturbed advancing front coordinate $\xi = \xi_e$ are:

$$s = s_0(\xi) + \varepsilon\bar{s}_1(\xi)\exp\left(\mathrm{i}\left(w_x x + w_y y\right) - kt\right) + O\left(\varepsilon^2\right), \tag{28}$$

$$\xi_e = 0 + \varepsilon\xi_e^1\exp\left(\mathrm{i}\left(w_x x + w_y y\right) - kt\right) + O\left(\varepsilon^2\right). \tag{29}$$

Substituting (28) into (4) we obtain the perturbed equation in the wetting region

$$\xi < 0: \; -k\bar{s}_1 = \frac{d^2}{d\xi^2}(D(s_0)\bar{s}_1) - \frac{d}{d\xi}((K'(s_0) - v)\bar{s}_1) - \omega^2 D(s_0)\bar{s}_1, \tag{30}$$

which is similar to equation (16) for the RE model but is valid only for the negative ξ. Substituting (29) into conditions on the wetting front equation (26) and equation (27) yields for $\xi = 0$:

$$\bar{s}_1 + \xi_e^1\frac{ds_0}{d\xi} = 0,$$

$$-k(s_e - s_+)\xi_e^1 = -\frac{d}{d\xi}(D(s_0)\bar{s}_1) - \xi_e^1\frac{d}{d\xi}\left(D(s_0)\frac{ds_0}{d\xi}\right).$$

Eliminating ξ_e^1 from these equations and accounting for (9), we get the following boundary condition for $\bar{s}_1$ to the perturbed flow equation (30)

$$\xi = 0: \quad k\left(\frac{s_e - s_+}{s_0'}\right)\bar{s}_1 = (K'(s_0) - v)\bar{s}_1 - \frac{d}{d\xi}(D(s_0)\bar{s}_1). \tag{31}$$

Replacing $\bar{s}_1$ by a new function Ψ:

$$\Psi = \sqrt{\frac{D(s_0)}{s_0'}}\bar{s}_1$$

and providing linearizing transformations, similar to that in Appendix B, to the perturbed problem (30) and (31), we get the following spectral problem for Ψ

$$\xi < 0: \quad -\Psi'' + \left(\omega^2 + F\right)\Psi = \frac{k}{D(s_0)}\Psi, \tag{32}$$

$$\xi = 0: \quad \Psi' = \left(\frac{B'}{B} - k\frac{s_e - s_+}{B^2}\right)\Psi, \tag{33}$$

where $F = B''/B$ and $B = \sqrt{D(s_0)s_0'}$. The appropriate spectrum technique consists of seeking for k for any fixed ω such that the problem (32) and (33) has finite nontrivial solutions.

5.2. Stability Analysis

[42] For convenience, we rewrite equation (32) considering k as a parameter of the spectral problem and $-\omega^2$ as an eigenvalue. Denoting $-\omega^2$ through λ, we arrive at the standard spectral problem for the Schrödinger operator

$$\xi < 0: \quad \Psi'' - U(k,\xi)\Psi = -\lambda\Psi$$

with the third-kind boundary condition (33). The potential $U(k, \xi) = F - k/D(s_0)$ is lower bounded for ξ, and therefore the operator of the problem is self-adjoint and lower semi-bounded in the space $L_2(-\infty, 0)$ of square integrable functions. This formulation allows us to apply standard techniques. The analysis presented in Appendix C demonstrates that for any $k \leq 0$ there exists eigenvalue $\lambda_0(k) < 0$ monotonically increasing with increase of k from $-\infty$ at $k = -\infty$ to zero at $k = 0$, and there are no other spectral points in the $\{\lambda \leq 0, k \leq 0\}$.

[43] Now we need to write these results for $\lambda(k)$ in terms of $k(\omega)$: For any ω there exists a unique negative eigenvalue $k_0(\omega)$ of the problem (32) and (33), and this value monotonically decreases with increase of ω:

$$k_0(0) = 0, \qquad k_0(\omega) \to -\infty \quad \text{as} \quad \omega \to \infty.$$

Any other part of the spectrum is positive.

[44] Negativeness of the eigenvalue $k_0(\omega)$ dictates that the traveling wave solution of the SFRE model is unstable, while monotonicity of $k_0(\omega)$ concludes an absence of preferentially growing perturbation modes meaning that the smaller the perturbation length $(2\pi/\omega)$, the faster the growth of such perturbations.

6. Nonequilibrium Richards Equation

[45] In this section we present a possible modification of the Richards equation and consider a family (or ensemble) of new models. For this family we maintain the mass balance equation (1) and modify the equilibrium pressure-saturation relationship (2). One type of family may be obtained by applying a dynamic capillary pressure theory (or dynamic memory effects) [*Gray and Hassanizadeh*, 1991]. The key point in the theory of dynamic memory effects is the replacing of the equilibrium relation (2) with a kinetic equation such as

$$\tau\dot{s} = p - \mathcal{P}(s), \tag{34}$$

where p represents a dynamic nonequilibrium pressure. In the following analysis we do not specify the form of the new rheological relationship which replaces equation (2), and use its general representation

$$F(s, p, \dot{s}, \dot{p}, \ddot{s}, \ddot{p}, \ldots) = 0 \tag{35}$$

instead of (34). The system of equations (1) and (35) is called the nonequilibrium Richards equation (NERE) model.

[46] We define $(s_0(\xi), p_0(\xi))$ as some traveling wave type solution for the NERE model with boundary conditions (7). If such solution exists then the wetting front velocity v is independent of equation (35) and designated by the formula (8). In the sections below we apply a linear stability analysis to examine this solution.

6.1. Stability Analysis

[47] Following the general scheme of the linear stability analysis we seek the perturbed solution of equations (1) and (35) in a form analogous to equation (25):

$$s = s_0(\xi) + \varepsilon S(\xi)\exp\left(\mathrm{i}\left(w_x x + w_y y\right) - kt\right) + O\left(\varepsilon^2\right), \tag{36}$$

$$p = p_0(\xi) + \varepsilon P(\xi)\exp\left(\mathrm{i}\left(w_x x + w_y y\right) - kt\right) + O\left(\varepsilon^2\right). \tag{37}$$

Substituting equations (36) and (37) into equations (1) and (35) leads to a system of two perturbation equations for the eigenvalues k and eigenfunctions S and P. One of these equations

$$\omega^2 K(s_0)P + \frac{dA}{d\xi} = kS \tag{38}$$

is arrived at from the mass balance equation (1), where

$$A = K'(s_0)\left(1 - \frac{dp_0}{d\xi}\right)S - vS - K(s_0)\frac{dP}{d\xi}$$

while the second equation written in functional form as

$$\mathcal{F}\left(S, P, S', P', \ldots; s_0, p_0, s_0', p_0', \ldots; k\right) = 0 \tag{39}$$

is governed by the rheological relation (35). We notice that there are no spatial derivatives in the equation (35) and it results in $\mathcal{F}$ being independent of both ω_x and ω_y. Boundary conditions for S and P require the flux A to vanish as $\xi \to \pm\infty$. Integrating equation (38) and using no flux boundary conditions yields

$$\omega^2 \int_{-\infty}^{+\infty} K(s_0)P\, d\xi = k\int_{-\infty}^{+\infty} S\, d\xi. \tag{40}$$

[48] The key moment in the following analysis is that for $\omega_x = \omega_y = 0$ it is possible to determine one of the eigenvalues ($k_0 = 0$) of the problem and the corresponding eigenfunctions ($S = s_0'$, $P = p_0'$). We note that the traveling wave solution for this modified Richards equation is invariant with regard to the arbitrary shift ε, as may be seen from equations (1) and (35) where time and spatial coordinates occur only in derivatives terms. This leads to $(s, p) = (s_0(\xi + \varepsilon), p_0(\xi + \varepsilon))$ considered as one of the possible perturbed solutions for the system equations (1) and (35). Expanding $s_0(\xi + \varepsilon)$ and $p_0(\xi + \varepsilon)$ into the Taylor series of ε

$$s = s_0(\xi) + \varepsilon s_0'(\xi) + O\left(\varepsilon^2\right), \tag{41}$$

$$p = p_0(\xi) + \varepsilon p_0'(\xi) + O\left(\varepsilon^2\right), \tag{42}$$

and comparing equations (41) and (42) with equations (36) and (37) at $k = 0$ and $\omega_x = \omega_y = 0$ we obtain that the eigenfunctions corresponding to $k_0 = 0$ are $s_0'(\xi)$ and $p_0'(\xi)$.

[49] Now we construct the asymptotic solution for k_0 for infinitesimal ω_x and ω_y. The frequencies ω_x and ω_y occur in the perturbation equation (38) only within the coefficient ω^2, and, then, we may expand eigenvalue k_0 and corresponding eigenfunctions by their expansion in power series of ω^2. It gives

$$k_0 = 0 + b\omega^2 + \ldots,\ S = s_0' + \tilde{S}\omega^2 + \ldots,\ P = p_0' + \tilde{P}\omega^2 + \ldots \tag{43}$$

Coefficient b determines the behavior of k_0 for low-frequency perturbations and may be found by substituting (43) into (40) and integrating the resulting equation with boundary conditions (7):

$$b = \frac{\mathcal{C}}{s_+ - s_-}, \quad \text{where} \quad \mathcal{C} = \int_{-\infty}^{+\infty} K(s_0)p_0'\, d\xi.$$

Stability of the perturbations corresponding to the eigenvalue k_0 for a small ω is designated by a sign of b which is opposite to the sign of integral $\mathcal{C}$, because of $s_+ < s_-$. This means that the low-frequency perturbation is stable for $\mathcal{C} < 0$ and unstable otherwise, i.e., $\mathcal{C} > 0$. It is known that the basic solution is considered unstable if the solution for the corresponding perturbed equation turns out to be unstable for at least one frequency eigenvalue. So, we have now established that if $\mathcal{C} > 0$ then the traveling wave solution of the NERE model equations (1) and (35) is unstable.

6.2. Low-Frequency Instability Criterion (LFC)

[50] In the previous section we state the stability criterion in terms of the sign of the coefficient $\mathcal{C}$. If an opposite criterion, such as if $\mathcal{C} < 0$ then traveling wave solution of the NERE equations (1) and (35) is stable, would be proved then the sign of $\mathcal{C}$ would give us a complete stability criterion. However, it seems to be impossible to prove this opposite criterion for the general form equation (35) of the rheological relationship. Therefore in this section we discuss this criterion (called LFC) for a specific case of low-frequency perturbations when $\omega \ll 1$. To provide the analysis we make the following assumption: for $\omega = 0$ the perturbation equation does not have negative eigenvalues with $k_0 = 0$ being a minimum eigenvalue. From the physical standpoint, this assumption says that only lateral perturbations may cause instability of the traveling wave solution, i.e., the traveling wave solution is stable if there is no lateral perturbation, or $\omega = 0$.

[51] The LFC is valid for the Richards equation, as may be seen from the analysis in section 4.3. For the case of the RE model the coefficient $\mathcal{C}$ is negative because pressure is a monotonic function for the traveling wave solution, and the LFC predicts the traveling wave solution to be stable. This result is valid not only for a small ω but for any perturbations as shown by the analysis given in section 3.

[52] The LFC may be used to describe stability for the relaxation model equation (34) as well. The result shown by *Cuesta et al.* [2000] is that the traveling wave solution for this model is monotonic for lower values of τ (smaller nonequilibrium effects), and is nonmonotonic for higher values of τ (larger nonequilibrium effects). For this case, the LFC predicts a transition from stability to instability of solutions with increase of τ. Numerical simulations provided by *Egorov et al.* [2002] confirm this conclusion.

[53] The LFC illustrates that instability of the traveling wave solution for a wide class of models having equations (1) and (35) as the general form (i.e., the NERE model) is caused by nonmonotonicity of the pressure distribution in the basic solution, and $\mathcal{C} > 0$ only if the function p_0 increases with depth within some interval of ξ. Therefore the NERE must produce nonmonotonicity in the basic (or traveling wave type) solution to be unstable and thereby be able to generate fingers. This requirement to the new hypothetical model matches the experimental results, for example, in the work by *Selker et al.* [1992], where physical nonmonotonicity was shown to be an essential characteristic of gravity-driven unstable flow.

7. Discussion

[54] To recap the results presented in this manuscript, we have shown by linear stability analysis that the Richards equation is unconditionally stable to infinitesimal perturbations. This was accomplished by applying a general perturbation to the basic solution. Symmetrizing the resulting perturbed equation by introducing a new space coordinate ξ, and function θ, led to an elliptic eigenvalue problem which was analyzed to provide stability criteria. The only assumption made in this analysis was that the conductivity-saturation function be convex. *Ursino* [2000] arrived at the same conclusion regarding unconditional stability of the Richards equation, while *Diment and Watson* [1983] hinted at such a conclusion in their work using a numerical solution for the perturbed flow equation.

[55] We also analyzed the general case of a finitely perturbed flow field in heterogeneous porous media, using an extension of the analyses presented by *Alt et al.* [1984] and *Otto* [1997]. The result of this analysis was that the Richards equation is unconditionally stable even for finite perturbations. No restriction was placed on the form of the conductivity-saturation function for this nonlinear analysis. These stability analyses are very general and very strong, and we conclude that the Richards equation is unconditionally stable, and therefore is not an appropriate equation to use in modeling unstable flows.

[56] This conclusion of unconditional stability of the Richards is in contrast to the conclusions derived by *Kapoor* [1996] and *Du et al.* [2001]. The reason for this difference lies in the simplifying assumptions used by these authors in making their mathematical problem statement tractable to solution. Those assumptions/limitations will now be discussed.

[57] The analysis by *Du et al.* [2001] started out with the basic solution in the same way the analysis was performed in this manuscript. In fact, their analysis is essentially correct up through equation (15) of their paper. The point of departure in their paper occurs where they began to analyze the perturbed flow equation. At that point they made simplifying assumptions of analyzing a perturbation initiating from a single point in the flow, leading to an equation with constant coefficients. This simplified equation does not adequately represent the full perturbed basic solution as analyzed in this manuscript. It is this result that led *Du et al.* [2001] to conclude incorrectly that the Richards equation is conditionally stable to infinitesimal perturbations.

[58] The stability analysis by *Kapoor* [1996] was based on the assumption of steady state flow, and determined the conditions for which infinitesimal perturbations to the steady state flow would amplify (flow becomes unstable). His mathematical formulation is correct up to equation (8). At that point in his formulation he simplifies his eigenvalue problem by assuming the planar perturbations to be small compared to the perturbations in the direction of flow. This assumption transforms the perturbation equation to a simple algebraic equation (equation (16)) for the growth factor. This simplification leads to the incorrect conclusion that the Richards equation is conditionally stable.

[59] The Richards equation is the well-accepted equation for representing the mass balance for flow in unsaturated porous media. The conclusion that this equation cannot represent unstable flows brings to question as to what should be the appropriate equation to represent such flows. Two modifications to the Richards equation were considered in this manuscript to derive a model for representing unstable flow. These will be discussed in the following.

[60] One modification to the RE model is that given by the Stefan-like SFRE model, which is based on the air entry pressure concept (section 5). Selecting this model was based on the observations in fingering experiments that the wetting front is very sharp and followed by a decrease in

saturation. While this modification to the RE does lead to instability of the traveling wave solution, we are very skeptical about using this model for fingering. The reason for the scepticism is that the function $k_0(\omega)$ is monotonic in the SFRE model and, hence, no fastest growing perturbation can be distinguished. The instability produced by the SFRE model is similar to the persistence-free Saffman-Taylor instability. It is known that unconditional instability produces a tree-like fractal structure because high-frequency perturbations have the fastest growth. However, experimentally observed fingers [e.g., *Glass et al.*, 1989; *Selker et al.*, 1992] have a well-defined width. This very distinguished discrepancy is not likely to be eliminated by introducing hysteretic properties in capillary pressure-saturation relationship. The dominating finger phenomenon must therefore be produced by another type of instability such as that described next.

[61] Based on discussions by others we perceive that fingering in unsaturated porous media is caused by processes that occur at the pore scale [*Ursino*, 2000, p. 270]. The conventional theory given by the RE is inadequate to represent fingering phenomena because the RE is derived by upscaling microscopic equations that do not account for these pore scale processes. Therefore some modification is needed to the macroscopic flow equations to incorporate the microscopic flow phenomena. We have suggested one possible model is that given by the nonequilibrium capillary pressure-saturation relation. That relation has been derived by either upscaling the microscale capillary phenomenon using homogenization [*Panfilov*, 1998] or applying thermodynamic considerations [*Hassanizadeh and Gray*, 1993].

[62] Adapting this upscaled result we modified the RE with a model containing dynamic memory effects. As shown in section 6, the traveling wave solution for this model must be nonmonotonic to produce instability. *Egorov et al.* [2002] have shown that the traveling wave solution for the NERE model using the special rheological function given by equation (34) [*Gray and Hassanizadeh*, 1991] turned out to be conditionally stable, and therefore appears to be an appropriate model to describe fingering. A comprehensive study of this NERE model is straightforward and will be considered in a subsequent paper.

Appendix A: Proof of Equation (5) for the Case of Heterogeneous Porous Media

[63] Let Ω be a bounded porous medium with a piecewise continuous boundary Γ and is divided on a set of subdomains Ω_i with distinctive geometry (properties). Richards' equations (1) and (3) hold within each subdomain holding the specific functions $K_i(s)$ and $\mathcal{S}_i(p)$. On the interfaces between subdomains pressure and normal water flux functions hold continuous. The outer boundary of Ω is divided on two parts: on the first part, Γ_p, we specify pressure, while on the second one, Γ_q, normal flux q_n to the boundary. Two possible solutions to the problem with two different initial conditions s_1^0 and s_2^0 are considered. Let $p_1, s_1 = \mathcal{S}(p_1)$ be the first solution, while p_2, $s_2 = \mathcal{S}(p_2)$ be the second possible solution. Hereafter we use the notation $\mathcal{S}(p)$ and $K(s)$ to define the functions in Ω and they coincide with $\mathcal{S}_\rangle$ and K_i in Ω_i respectively. Let us also define the hydraulic conductivity $\mathcal{K}(p) = K(\mathcal{S}(p))$ as a function of pressure obeying the following inequality

$$|\mathcal{K}(p_a) - \mathcal{K}(p_b)| \leq A|p_a - p_b|, \quad \forall\, p_a, p_b \tag{A1}$$

with a constant A being unique within all subdomains. Since functions $\mathcal{S}_\rangle$ and K_i are smooth the constant A may be taken as

$$A = \max_i \left(\max_s K_i'(s) \cdot \max_p \mathcal{S}_i'(p) \right).$$

[64] Multiplying equation (1) by some continuous and smooth function η, with the condition $\eta = 0$ on Γ_p, and then integrating equation (1) over Ω by parts yields

$$\int_\Omega \frac{\partial s_1}{\partial t} \eta d\Omega + \int_\Omega \mathcal{K}(p_1)(\nabla p_1 + \vec{e}_z)\nabla\eta d\Omega = \int_{\Gamma_q} q_n \eta d\Gamma_q, \tag{A2}$$

where q_n is the flux normal to the boundary. An identity analogous to equation (A2) may be also derived for p_2. Subtracting one of these identities from another we get

$$\begin{aligned} &\int_\Omega \frac{\partial(s_1 - s_2)}{\partial t} \eta d\Omega = J - J_1, \\ &J_1 = \int_\Omega \mathcal{K}(p_1)\nabla(p_1 - p_2)\cdot\nabla\eta d\Omega, \\ &J = \int_\Omega (\mathcal{K}(p_2) - \mathcal{K}(p_1))(\nabla p_2 + \vec{e}_z)\cdot\nabla\eta d\Omega. \end{aligned} \tag{A3}$$

The following analysis relies on choosing the form of the function η in equation (A3).

[65] We introduce an auxiliary continuously-differentiable function μ: $(-\infty, \infty) \to [0,1]$, such as (1) $\mu(\xi) \equiv 0$ for $\xi \leq 0$, (2) $\mu(\xi) \equiv 1$ for $\xi \geq 1$, and (3) $\mu(\xi)$ monotonically increases with ξ for $0 < \xi < 1$. We also introduce a parametric family of functions $\eta_\varepsilon(\xi) = \mu(\xi/\varepsilon)$ based on the function $\mu(\xi)$ and bearing two main asymptotic properties

$$\eta_\varepsilon(\xi) \to \mathrm{sign}_0(\xi) = \begin{cases} 0 & \text{for } \xi \leq 0, \\ 1 & \text{for } \xi > 0, \end{cases} \quad \text{as} \quad \varepsilon \to 0, \tag{A4}$$

$$B(\varepsilon) = \max_\xi \left(\xi^2 \eta_\varepsilon'(\xi)\right) \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{A5}$$

Now we pick the form of η in equation (A3) as $\eta = \eta_\varepsilon(p_1 - p_2)$. The requirement that the functions p_1 and p_2 coincide on Γ_p forces the condition $\eta = 0$ being performed on this part of the boundary.

[66] We now estimate the right-hand side of equation (A3). We recall that $\nabla\eta_\varepsilon(p_1 - p_2) = \eta'_\varepsilon(p_1 - p_2)\,\nabla(p_1 - p_2)$ and it allows us to transform J_1 and J in equation (A3) to

$$J_1 = \int_\Omega \mathcal{K}(p_1)|\nabla(p_1 - p_2)|^2 \eta_\varepsilon'(p_1 - p_2) d\Omega,$$

$$J = \int_\Omega (\mathcal{K}(p_2) - \mathcal{K}(p_1))(\nabla p_2 + \vec{e}_z)\cdot\nabla(p_1 - p_2)\eta_\varepsilon'(p_1 - p_2) d\Omega.$$

Using equation (A1) and applying the Cauchy-Schwartz inequality in conjunction with definition equation (A5) for B we estimate J as

$$|J| \leq \int_\Omega A|p_1 - p_2|(|\nabla p_2| + 1)|\nabla(p_1 - p_2)|\eta'_\varepsilon(p_1 - p_2)d\Omega$$

$$\leq \sqrt{J_1 J_2},$$

$$J_2 = A^2 \int_\Omega (p_1 - p_2)^2 \eta'_\varepsilon(p_1 - p_2)\frac{(|\nabla p_2| + 1)^2}{\mathcal{K}(p_1)}d\Omega$$

$$\leq A^2 B(\varepsilon)\int_\Omega \frac{(|\nabla p_2| + 1)^2}{\mathcal{K}(p_1)}d\Omega.$$

[67] The combination of J_1 and J_2 produces $\sqrt{J_1 J_2} \leq J_1 + J_2/4$, which then yields the necessary estimation

$$J - J_1 \leq \frac{1}{4}A^2 B(\varepsilon)\int_\Omega \frac{(|\nabla p_2| + 1)^2}{\mathcal{K}(p_1)}d\Omega.$$

[68] Having this estimate we obtain

$$\int_\Omega \frac{\partial(s_1 - s_2)}{\partial t}\eta_\varepsilon(p_1 - p_2)d\Omega \leq \frac{1}{4}A^2 B(\varepsilon)\int_\Omega \frac{(|\nabla p_2| + 1)^2}{\mathcal{K}(p_1)}d\Omega. \tag{A6}$$

The integral at the right-hand side of equation (A6) is bounded, and tending ε to zero and using equations (A4) and (A5), we can transform equation (A6) by integrating it with time to

$$\int_0^t \int_\Omega \frac{\partial(s_1 - s_2)}{\partial t}\mathrm{sign}_0(p_1 - p_2)d\Omega dt \leq 0. \tag{A7}$$

The subdomain of $\Omega \times [0, t]$ where $s_1 \equiv 1$ and $s_2 \equiv 1$ simultaneously contributes zero in integral (A7) because saturation holds constant in saturated media and both $\partial s_1/\partial t$ and $\partial s_2/\partial t$ equal zero. In the rest of $\Omega \times [0,t]$, either s_1 or s_2 is less than a unity and $\mathrm{sign}_0(p_1 - p_2) = \mathrm{sign}_0(s_1 - s_2)$ because function $\mathcal{S}(p)$ is monotonic. This shows that $\mathrm{sign}_0(p_1 - p_2)$ in equation (A7) may be replaced by $\mathrm{sign}_0(s_1 - s_2)$ transforming equation (A7) to

$$\int_0^t \int_\Omega \frac{\partial}{\partial t}(s_1 - s_2)^+ d\Omega dt \leq 0, \tag{A8}$$

where the superscript + indicates the positive part of the function, i.e., $s^+ = \max(0, s)$. Equation (A8) is also valid if solutions s_1 and s_2 are exchanged. Thus using elementary equality $|s| = (s)^+ + (-s)^+$ yields

$$\int_0^t \int_\Omega \frac{\partial}{\partial t}|s_1 - s_2| d\Omega dt \leq 0.$$

Finally, taking the time derivative out of the integral proves inequality (5).

Appendix B: Transformation of Perturbed Equation for the RE Model

[69] Introducing new function $\tilde{s}_1$, such as $\bar{s}_1 = -s'_0\tilde{s}_1$, we rewrite equation (16) as

$$\frac{\partial\tilde{s}_1}{\partial t} + \omega^2 D(s_0)\tilde{s}_1 + \frac{d\xi}{ds_0}\frac{\partial F}{\partial\xi} = 0,$$

$$F = (K'(s_0) - v)s'_0\tilde{s}_1 - \frac{\partial}{\partial\xi}\left(D(s_0)s'_0\tilde{s}_1\right).$$

Identically transforming F by

$$F = \frac{d}{d\xi}\left(K(s_0) - vs_0 - D(s_0)\frac{ds_0}{d\xi}\right)\tilde{s}_1 - D(s_0)s'_0\frac{\partial\tilde{s}_1}{\partial\xi}$$

and noticing that the first term in the right-hand side is equal to zero by virtue of equation (9), we obtain

$$\frac{\partial\tilde{s}_1}{\partial t} + \omega^2 D(s_0)\tilde{s}_1 - \frac{d\xi}{ds_0}\frac{\partial}{\partial\xi}\left(D(s_0)\frac{ds_0}{d\xi}\frac{\partial\tilde{s}_1}{\partial\xi}\right) = 0. \tag{B1}$$

Replacing ξ and $\tilde{s}_1$ by ζ and θ respectively defined in equation (17) where $\theta = B\tilde{s}_1$ we obtain from equation (B1)

$$\frac{\partial\theta}{\partial t} + \omega^2 D\theta = \frac{1}{B}\frac{\partial}{\partial\zeta}B^2\frac{\partial}{\partial\zeta}\left(\frac{\theta}{B}\right). \tag{B2}$$

Simple transformations shows that the right-hand side of equation (B2) is equal to

$$\frac{\partial^2\theta}{\partial\zeta^2} - \frac{\theta}{B}\frac{\partial^2 B}{\partial\zeta^2}$$

which yields the perturbed equation (18).

Appendix C: Analysis of Spectral Problem for the SFRE Model

[70] The potential U being bounded requires that the essential spectrum of the problem is the interval $[\lambda_{ess}(k), \infty]$ with

$$\lambda_{ess}(k) = \lim_{\xi\to-\infty} U(k,\xi) = F_- - k/D_-.$$

Notice that since $F_- = ((K'_- - v)/2D_-)^2$ and D_- are positive this essential spectrum lies outside the quadrant $\{\lambda \leq 0, k \leq 0\}$.

[71] Eigenvalues $\lambda_0(k)$, $\lambda_1(k)$,... lie lower than the essential spectrum, and $\lambda_0(k) < \lambda_1(k) < \, < \lambda_{ess}(k)$. Eigenfunctions Ψ corresponding to these eigenvalues belong to a space V of functions such as squares of both these functions and their first derivatives are integrable on $[0,\infty]$. It follows that the eigenvalues and corresponding eigenfunctions may be found by the variational Courant-Fisher principle

$$\lambda_n(k) = \min_{V_{n+1}\subset V}\max_{\psi\in V_{n+1}}(I_0(\psi) + kI_1(\psi))/|\psi|^2, \; n = 0, 1, \ldots \tag{C1}$$

where V_n is the n-dimensional subspace V, $|\psi|^2 = \int_{-\infty}^{0} \psi^2 d\xi$ is the norm square in $L_2(-\infty,0)$, and functionals I_0 and I_1 are

$$I_0(\psi) = \int_{-\infty}^{0} \left(\psi'^2 + F\psi^2\right) d\xi - \frac{B'}{B}\psi^2(0),$$

$$I_1(\psi) = -\int_{-\infty}^{0} \frac{1}{D}\psi^2 \, d\xi + \frac{s_e - s_+}{B^2(0)}\psi^2(0).$$

[72] Let us study the main eigenvalue of the problem $\lambda_0(k)$. The Courant-Fisher principle may be written for the main eigenvalue as

$$\lambda_0(k) = \min_{\psi \in V} (I_0(\psi) + kI_1(\psi))/|\psi|^2. \quad \text{(C2)}$$

Substitution of $k = 0$ into equation (C2) concludes that function $\Psi = B$ turns out to be the eigenfunction corresponding to the eigenvalue $\lambda = 0$ ($\omega = 0$). Since $B > 0$, this eigenvalue is considered as the main eigenvalue and may be derived from (C2) as

$$\lambda_0(0) = \min_{\psi \in V} I_0(\psi)/|\psi|^2 = I_0(B)/|B|^2 = 0. \quad \text{(C3)}$$

Let us choose B as a sample function for equation (C2). Using (C3) we obtain that $\lambda_0(k) \leq kI_1(B)/|B|^2$. $I_1(B)$ is expressed explicitly as

$$I_1(B) = -\int_{-\infty}^{0} s_0' \, d\xi + (s_e - s_+) = (s_- - s_+),$$

which yields the following estimation of the main eigenvalue

$$\lambda_0(k) \leq ak, \quad a = (s_- - s_+)/|B|^2 > 0. \quad \text{(C4)}$$

This upper limit for $\lambda_0(k)$ in conjunction with the property of the continuous spectrum shown above demonstrates that $\lambda_0(k)$ exists for any negative k. Function $\lambda_0(k)$ is continuous and convex. The latter property is arrived at due to the Rayleigh relation in (C2) being linear with regard to k. This finally establishes that λ_0 monotonically increases with increase of k from $-\infty$ at $k = -\infty$ to zero at $k = 0$.

[73] Now we show that the remaining part of the spectrum lies outside the quadrant $\{\lambda \leq 0,\ k \leq 0\}$. This statement has been proved above for the essential part of the spectrum. For the discrete part this holds because (1) $\lambda_n(0) > \lambda_0(0) = 0$ for all $n \geq 1$ and (2) $\lambda_n(k) > 0$ is valid for $k < 0$ and any $n \geq 1$. To prove this we designate V_n^0 to the subspace of functions ψ from V_n and satisfying the condition $\psi(0) = 0$. Using equation (C3) the inequality

$$I_0(\psi) + kI_1(\psi) \geq I_0(B) - k\int_{-\infty}^{0} \frac{1}{D}\psi^2 \, d\xi \geq b|\psi|^2$$

with positive $b = -k/D_-$ holds for $\psi \in V_{n+1}^0$ and $k < 0$, and therefore we get from equation (C1) the following chain of inequalities

$$\lambda_n(k) = \min_{V_{n+1} \subset V} \max_{\psi \in V_{n+1}} (I_0(\psi) + kI_1(\psi))/|\psi|^2$$
$$\geq \min_{V_{n+1} \subset V} \max_{\psi \in V_{n+1}^0} (I_0(\psi) + kI_1(\psi))/|\psi|^2 \geq b > 0. \quad \text{(C5)}$$

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